

WHAT IS QUANTUM THEORY OF GRAVITY?

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Abstract

We present a line by line derivation of canonical quantum mechanics stemming from the compatibility of the statistical geometry of distinguishable observations with the canonical Poisson structure of Hamiltonian dynamics. This viewpoint can be naturally extended to provide a conceptually novel, non-perturbative formulation of quantum gravity. Possible observational implications of this new approach are briefly mentioned.

1 What is Quantum Mechanics?

In this talk we would like to 1) give a line by line derivation of canonical quantum mechanics (QM) founded on the compatibility of the statistical geometry of distinguishable observations with the canonical Poisson structure of Hamiltonian dynamics and 2) describe a natural extension of this viewpoint so as to provide a conceptually novel approach to the problem of a non-perturbative formulation of quantum gravity, one which should reduce in the correspondence limit to general relativity (GR) coupled to matter degrees of freedom. The presentation is based on our recent work [1].

To understand the fundamental structure of QM we reason as follows: Assume that individual quantum events are statistical and statistically distinguishable (Postulate I). (This premise is of course a huge conceptual leap in comparison to classical physics, but it is absolutely crucial for the structure of QM.) On the space of probability distributions there is a natural metric, called Fisher metric, which provides a geometric measure of statistical distinguishability [2]

$$ds^2 = \sum_i \frac{dp_i^2}{p_i}, \quad \sum_i p_i = 1, \quad p_i \geq 0. \quad (1)$$

(This distance naturally arises as follows: to estimate probabilities p_i from frequencies f_i , given N samples, when N is large, the central limit theorem states that the probability for the frequencies is given by the Gaussian distribution $\exp(-\frac{N}{2} \frac{(p_i - f_i)^2}{p_i})$. Thus a probability distribution $p_i^{(1)}$ can be distinguished from a given distribution p_i provided the Gaussian $\exp(-\frac{N}{2} \frac{(p_i^{(1)} - p_i)^2}{p_i})$ is small. Hence the quadratic form $\frac{(p_i^{(1)} - p_i)^2}{p_i}$, or its infinitesimal form

$\sum_i \frac{dp_i^2}{p_i}$, the Fisher distance, is the natural measure of distinguishability.) Now, upon setting $p_i = x_i^2$, making p_i manifestly non-negative, the Fisher distance reads $ds^2 = \sum_i dx_i^2$ with $\sum x_i^2 = 1$ i.e. the Euclidean metric on a sphere. Therefore the latter distance is nothing but the shortest distance along this unit sphere [2]

$$ds_{12} = \cos^{-1}(\sum_i \sqrt{p_{1i}}\sqrt{p_{2i}}). \quad (2)$$

Next, we demand that on this metric space of probabilities one can define a canonical Hamiltonian flow (Postulate II). So the dimensionality of such a symplectic space of x_i must be *even* (hence, $\sum x_i^2 = 1$ defines an odd-dimensional sphere). Then the Hamiltonian flow is given (locally) as

$$\frac{df(x_i)}{dt} = \omega_{ij} \frac{\partial h(x_i)}{\partial x_i} \frac{\partial f(x_i)}{\partial x_j} \equiv \{h, f\}, \quad (3)$$

where ω is a closed non-degenerate 2-form. The compatibility of the symplectic form ω and the metric g allows for the introduction of an almost complex structure $J \equiv \omega g^{-1}$ (in the matrix notation), $J^2 = -1$ (since the compatibility demands $\omega_{ij} g^{jk} \omega_{kl} = g_{il}$). Given this constant complex structure we may introduce complex coordinates on this space ψ_a (and their conjugates ψ_a^*), so that $\sum_i x_i^2 \equiv \sum_a \psi_a^* \psi_a = 1$, and thus $p_a = \psi_a^* \psi_a$. This statistical distance is invariant under $\psi \rightarrow e^{J\alpha} \psi$, J being the above integrable almost complex structure. Thus ψ can be identified with $e^{J\alpha} \psi$. Indeed an odd dimensional sphere can be viewed as a $U(1)$ Hopf-fibration of a complex projective space $CP(n)$, a coset space $\frac{U(n+1)}{U(n) \times U(1)}$. $CP(n)$ is a homogeneous, isotropic and simply connected Kahler manifold with a constant holomorphic sectional curvature. The unique metric on $CP(n)$ is the Fubini-Study (FS) metric (which is but the above statistical Fisher metric up to a multiplicative constant, the Planck constant \hbar). In the Dirac notation (using (2) and the derived Born rule, $p_a = \psi_a^* \psi_a$), this FS metric reads :

$$ds_{12}^2 = 4(\cos^{-1}|\langle \psi_1 | \psi_2 \rangle|)^2 = 4(1 - |\langle \psi_1 | \psi_2 \rangle|^2) \equiv 4(\langle d\psi | d\psi \rangle - \langle d\psi | \psi \rangle \langle \psi | d\psi \rangle), \quad (4)$$

Thus $CP(n)$ is the underlying manifold of statistical events with a well defined Hamiltonian flow and as such provides a kinematical background on which a Hamiltonian dynamics is defined. The only Hamiltonian flow compatible with the isometries of $CP(n)$ (which are the unitaries $U(n+1)$) is given by a quadratic function of x_i or, alternatively, a quadratic form in the pair $q_a \equiv \text{Re}(\psi_a)$ and $p_a \equiv \text{Im}(\psi_a)$, $h = \frac{1}{2} \sum_a [(p_a)^2 + (q_a)^2] \omega_a$, or in the usual notation, $h = \langle \hat{H} \rangle$, ω_a being the eigenvalues of \hat{H} . The Hamiltonian equations for the ψ_a and its conjugate then give the linear evolution equation (Schrödinger equation), $\frac{dp_a}{dt} = \{h, p_a\}$, $\frac{dq_a}{dt} = \{h, q_a\}$, that is $J \frac{d|\psi\rangle}{dt} = H|\psi\rangle$. Any observable, consistent with the isometries of the underlying space of events, is given as a quadratic function in the q_a, p_a . These are just the usual expectation values of linear operators.

Everything we know about quantum mechanics is in fact contained in the geometry of $CP(n)$ [3], [4]: entanglements come from the embeddings of the products of two complex projective spaces in a higher dimensional one; geometric phases stem from the symplectic structure of $CP(n)$, quantum logic, algebraic approaches to QM etc, are all contained in the geometric and symplectic structure of complex projective spaces [3], [4]. (While we

only consider here the finite dimensional case, the same geometric approach is extendable to generic infinite dimensional quantum mechanical systems, including field theory.) Finally, the following three lemmas are important for the material of the next section: (A) The Fisher-Fubini-Study quantum metric (4) in the $\hbar \rightarrow 0$ limit becomes a spatial metric provided the configuration space for the quantum system under consideration *is* space. For example, consider a particle moving in 3-dimensional Euclidean space. Then the quantum metric for the Gaussian coherent state $\psi_l(x) \sim \exp(-\frac{(\vec{x}-\vec{l})^2}{\delta l^2})$ yields the natural metric in the configuration space, in the $\hbar \rightarrow 0$ limit, $ds^2 = \frac{d\vec{l}^2}{\delta l^2}$. (B) Similarly, the time parameter of the evolution equation can be related to the quantum metric via

$$\hbar ds = \Delta E dt, \quad \Delta E^2 \equiv \langle \psi H^2 \psi \rangle - (\langle \psi H \psi \rangle)^2 \quad (5)$$

(C) Finally, the Schrodinger equation can be viewed as a geodesic equation on a $CP(n) = \frac{U(n+1)}{U(n) \times U(1)}$

$$\frac{du^a}{ds} + \Gamma_{bc}^a u^b u^c = \frac{1}{2\Delta E} Tr(HF_b^a) u^b. \quad (6)$$

Here $u^a = \frac{dz^a}{ds}$ where z^a denote the complex coordinates on $CP(n)$, Γ_{bc}^a is the connection obtained from the FS metric, and F_{ab} is the canonical curvature 2-form valued in the holonomy gauge group $U(n) \times U(1)$.

The above geometric structure describing canonical QM, beautifully tested in numerous experiments, is also very robust from the purely geometric point of view [1]. A consistent generalization of QM would doubtlessly be interesting from both the experimental and theoretical viewpoints. Unlike various generalizations proposed in the past (which in many instances have lead to difficult conceptual problems) the one put forward in [1] extends the kinematical structure so that it is compatible with the generalized dynamical structure! *The quantum symplectic and metric structure, and therefore the almost complex structure become fully dynamical.* The underlying physical reason for such a more general dynamical framework of the above geometric formalism is found in the need for a quantum version of the equivalence principle, a fundamental physical underpinning of a non-perturbative formulation of quantum theory of gravity.

2 And What is Quantum Theory of Gravity?

The main intuition behind a quantum version of the equivalence principle is to demand the validity of the canonical QM, as laid out in the previous section, in every local neighborhood, *in the space of quantum events*. Here we envision a larger geometric structure whose tangent spaces, viewed as vector spaces by definition, are just canonical Hilbert spaces. (In this picture [1] the tangent *spatial* transverse metric emerges from the *quantum* metric, as in the lemma (A), by assuming that the underlying configuration space is space and time appears as a measure of the geodesic distance in this general space of statistical events (the events do not have to be necessarily distinguishable!), as in lemma (B). Finally, the longitudinal spatial coordinate corresponds to the dimensionality of the tangent Hilbert spaces.) The crucial point is to allow for any metric and symplectic form in the geodesic version of the evolution equation (lemma (C)). These are in turn determined dynamically [1].

Our postulates (I) and (II) as stated above can indeed be naturally extended by allowing both the metric and symplectic form on the space of quantum events to be no longer rigid but fully dynamical entities. In this process, just as in the case of spacetime in GR, the space of quantum events becomes dynamical and only individual quantum events make sense observationally. Specifically, we do so by relaxing the first postulate to allow for *any* statistical (information) metric while insisting on the compatibility of this metric with the symplectic structure underlying the second postulate. Physics is therefore required to be diffeomorphism invariant in the sense of information geometry such that the information geometric and symplectic structures remain compatible, requiring only a *strictly* (i.e non-integrable) almost complex structure J . This extended framework readily implies that the wave functions labeling the event space, while still unobservable, are no longer relevant. They are in fact as meaningless as coordinates in GR. There are no longer issues related to reductions of wave packets and associated measurement problems. At the basic level of our scheme, there are only dynamical correlations of quantum events. The observables are furnished by diffeomorphism invariant quantities in the space of quantum events.

To find the kinematical arena for this generalized framework we seek as coset of $Diff(C^{n+1})$ such that locally the latter looks like $CP(n)$ and allows for a compatibility of its metric and symplectic structures, expressed in the existence of a (generally non-integrable) almost complex structure. The following nonlinear Grassmannian

$$Gr(C^{n+1}) = Diff(C^{n+1})/Diff(C^{n+1}, C^n \times \{0\}), \quad (7)$$

with $n = \infty$ fulfills these requirements [5]. $Gr(C^{n+1})$ is a nonlinear analog of a complex Grassmannian since it is the space of (real) co-dimension 2 submanifolds, namely a hyperplane $C^n \times [0]$ passing through the origin in C^{n+1} . Its holonomy group $Diff(C^{n+1}, C^n \times \{0\})$ is the group of diffeomorphisms preserving the hyperplane $C^n \times \{0\}$ in C^{n+1} . Just as $CP(n)$ is a co-adjoint orbit of $U(n+1)$, $Gr(C^{n+1})$ is a coadjoint orbit of the group of volume preserving diffeomorphisms of C^{n+1} . As such it is a symplectic manifold with a canonical Kirillov-Kostant-Souriau symplectic two-form Ω which is closed ($d\Omega = 0$) but not exact. Indeed the latter 2-form integrated over the submanifold is nonzero; its de Rham cohomology class is integral. This means that there is a principal 1-sphere, a $U(1)$ or line bundle over $Gr(C^{n+1})$ with curvature Ω . This is the counterpart of the $U(1)$ -bundle of S^{2n+1} over $CP(n)$ of quantum mechanics. It is also known that there is an almost complex structure given by a 90 degree rotation in the two dimensional normal bundle to the submanifold. While $CP(n)$ has an integrable almost complex structure and is therefore a complex manifold, in fact a Kahler manifold, this is *not* the case with $Gr(C^{n+1})$. Its almost complex structure J is strictly *not* integrable in spite of its formally vanishing Nijenhuis tensor. While the vanishing of the latter implies integrability in the finite dimensional case, such a conclusion no longer holds in the infinite dimensional setting. However what we do have in $Gr(C^{n+1})$ is a strictly (i.e. non-Kahler) almost Kahler manifold since there is by way of the almost structure J a compatibility between the closed symplectic 2-form Ω and the Riemannian metric g which *locally* is given by $g^{-1}\Omega = J$. Clearly, it would be very interesting to understand how unique is the structure of $Gr(C^{n+1})$.

Just as in standard geometric QM, the probabilistic interpretation should be found in the definition of geodesic length on the new space of quantum states/events. Notably

since $Gr(C^{n+1})$ is only a strictly almost complex, its J is only locally complex. This fact translates into the existence of only local time and local metric on the space of quantum events. The local temporal evolution equation is a geodesic equation on the space of quantum events $\frac{du^a}{d\tau} + \Gamma_{bc}^a u^b u^c = \frac{1}{2E_p} Tr(HF_b^a) u^b$ where now τ is given through the metric $\hbar d\tau = 2E_p dt$, where E_p is the Planck energy. Γ_{bc}^a is the affine connection associated with this general metric g_{ab} and F_{ab} is a general curvature 2-form in the holonomy gauge group $Diff(C^{n+1}, C^n \times \{0\})$. This geodesic equation follows from the conservation of the energy-momentum tensor $\nabla_a T^{ab} = 0$ with $T_{ab} = Tr(F^{ac} g_{cd} F^{cb} - \frac{1}{4} g_{ab} F_{cd} F^{cd} + \frac{1}{2E_p} H u_a u_b)$. Since both the metrical and symplectic data are also contained in H and are $\hbar \rightarrow 0$ limits of their quantum counterparts [4], [1], we have here a non-linear “bootstrap” between the space of quantum events and the generator of dynamics. The diffeomorphism invariance of the new quantum phase space is explicitly taken into account in the following dynamical scheme [1]:

$$R_{ab} - \frac{1}{2} g_{ab} R - \lambda g_{ab} = T_{ab} \quad (8)$$

($\lambda = \frac{n+1}{\hbar}$ for $CP(n)$; in that case $E_p \rightarrow \infty$). Moreover we demand for compatibility

$$\nabla_a F^{ab} = \frac{1}{E_p} H u^b. \quad (9)$$

The last two equations imply via the Bianchi identity a conserved energy-momentum tensor, $\nabla_a T^{ab} = 0$. The latter, taken together with the conserved “current” $j^b \equiv \frac{1}{2E_p} H u^b$, i.e. $\nabla_a j^a = 0$, results in the generalized geodesic Schrödinger equation. As in GR it will be crucial to understand both the local and global features of various solutions to the above dynamical equations. The kinematical structure of ordinary QM is compatible with our general dynamical formulation and thus we naturally expect that our general formalism is compatible with all known cases in which quantum theories of gravity have been non-perturbatively defined, albeit in *fixed* asymptotic backgrounds (such as string theory in asymptotically AdS spaces).

What determines the form of the Hamiltonian H ? The only requirement is that H should define a canonical quantum mechanical system whose configuration space *is* space and whose dynamics defines a consistent quantum gravity in an asymptotically flat background. We are aware of only one example satisfying this criterion: Matrix theory [6]! *Thus our proposal defines a background independent, non-perturbative, holographic formulation of Matrix theory* [1]. This choice of the Hamiltonian might at first appear artificial given the generality of our proposal. Yet we note that, from the suggestive geodesic form of the Schrödinger equation, H can be viewed as a “charge”, and thus may be determined in a quantum theory of gravity by being encoded in the non-trivial symplectic topology of the space of quantum events. Such a realization may well be possible here since our non linear Grassmannian is non-simply connected [5].

Finally, among the possible observational implications of our proposal, those we are currently attempting to understand are: 1) possible deviations from the Planck law involving primordial gravitational waves, or Hawking radiation; 2) deviations from the classic QM formula for the vacuum energy (which underlies the cosmological constant problem); 3) relativity of the double slit experiment: once we relax postulate I, so that any information

metric is allowed, the relativity (observer dependence) of canonical QM experiments (such as the double-slit experiment) becomes possible; 4) highly constrained deviations from linearity (superposition principle), and canonical QM entanglement; 5) the fact that in our proposal the generalized geometric phase is in $Diff(C^{n+1}, C^n \times \{0\})$, is also amenable to experimental tests.

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